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Research article

EXISTENCE AND UNIQUENESS THEOREM FOR PERIODIC SOLUTIONS OF DELAY INTEGRO-DIFFERENTIAL EQUATION

Nwaoburu, A.O

Department of Mathematics, Faculty of Science,
Rivers State University of Science and Technology, P.M.B. 5080,
Nkpolu- Oroworukwo, Port Harcourt, Rivers State, Nigeria.
e-mail: nwaoburu.adols@ust.edu.ng

Abstract

Necessary and sufficient conditions for the existence and uniqueness of periodic solutions of delay integro-differential equations are proved here. The method of proof is based on the assumption of non-criticality of the homogenous part of the system which utilisation, furnishes a variation of constant formula for the system. A green matrix for the system is defined which converts the solution to its most precise term. Utilising the regularity and boundedness properties of the green matrix, a completely continuous operation is obtained and by Schauder's fixed point theorem has a fixed point which is the periodic solution of the integro-differential equation.

Keywords: existence, uniqueness, periodic solution, delay and integro-differential equation

1. Introduction

Conditions for the existence and uniqueness of solutions of differential equations are of immense importance in the field of Mathematics, Science, Engineering, etc. It constitutes the core contents, researches in the subject have been intensive and vast literature abound on the existence and uniqueness of solutions of ordinary differential equations with initial conditions given by:

$$y = f(x, y);\ y(x_0) = y_0 ----------------(1.1)$$

But there are scanty research results on delay systems. Recently, Peano, Picard as reported in Goddington and Levinson ( ), Cruz and Hale ( ) among others have provided conditions for the existence of solutions of systems (1.1); the continuity condition on f and its satisfaction of Lipshitz conditions. These conditions have been improved upon to provide the Caratheodory's condition for the existence and uniqueness of solutions of delay systems as disclosed by Hale ( ).

Interest on the subject has diversified to accommodate investigations on the existence and uniqueness of systems with boundary conditions and systems with periodic solutions. Recently, results on existence and uniqueness of solutions or ordinary differential equations have been carried over to functional differential systems by a host of scholars among whom are:

Chukwu ( ), Cruz and Hale ( ), Driver ( ), Hale ( ) provided sufficient conditions for the global existence and the exponential estimates of solutions of non-linear delay system of the form:

$$x(t) = L(t, x_t) + b(t);\ t > \delta --------------(1.2)$$

$$x_t = \phi(t);\ and\ t \in [\phi, \infty]$$

conditions for the existence and uniqueness of neutral system have been investigated in Driver ( ), Hale ( ). The generalises neutral functional inclusion given by:

$$\left[\frac{d}{dt}\right] D(t, x_t) \in R(t, x_t) ----------------(1.3)$$

was studied by Chukwu ( ) who placed certain regularity conditions on D and R and invoking K. Fan's fixed point theorem, proved the existence of solution of equation (1.3) which satisfies two point boundary values.

$$x_{t0}\,\phi\ and\ x_{t1} = \phi I$$

Recently Chukwu ( ) inspired by the studies of Roseau ( ), Rouche and Mawhin ( ) has provided conditions for the existence of the solution of a parameter dependent system.

$x = A(t)x + B(t)U + f(t,x,U);$
$t \in [t_0, t_0 + T]$ ------------------------(1.4)

With boundary conditions $x(t_0) = x(t_0 + T_0)$. From Chukwu ( ) and Hale ( ) we have an understanding of periodic systems and periodic solutions. The system

$x(t) = f(t, xt);\ xt_0 = \phi$ ------------------(1.5)

such that $f(t+w) = f(t)$ $for\ w > 0$ is called a periodic system. Also if $x(t)$ is a solution such that $x(t+w) = x(t),$ then the solution is said to be *w-periodic*.

Cruz and Hale( ), Chukwu ( ) have established a strong relationship between the boundedness of a solution, its continuation and periodicity when they found out that if the solutions of (1.3) are uniformly, ultimately bounded, then there exists periodic solutions.

Tejumola ( ) has drawn attention to the application of periodic systems and solutions to the theory of oscillation, resonance and non-resonance systems.

Iheagwam ( ) has pointed out that periodic solutions provide classical examples of boundary-value problems. He has shown that if a system has a non-critical linear part, its solutions would be periodic. The results of Ukwu ( ) and Iheagwam ( ) would be extended to integro-differential equations with linear delay base given by:

$x(t) = A(t)x(t) + B(t)x(t-1) + \int_{-1}^{0} k(t_1 s)x(t+s)ds$ and its perturbation also given by:

$$x(t) = A(t)x(t) + B(t)x(t-1) + \int_{-1}^{0} k(t_1 s)x(t+s)ds + f(t_1 x_{tt})$$

This is the main objective of this study.

2 Notation and Preliminaries

Let *E* denote the real line. For a real integar *n*, E^n denotes the space of real n-triples with the usual Euclidean norm 1.1 and $C([a,b], E^n)$ is the Banach Space of continuous functions from *[a, b]* into E^n with the supremum norm

$$\|\phi\| = \underset{-1<s<0}{Sup}|\phi(s)|\ for - \phi \in C([a,b], E^n) \text{--------------}(2.1)$$

in this work, the state space will be $C([-1,0], E^n)$ with norm defined by (2.1)

For the function $x:[-h,t]\rightarrow E^n$ and $t\in[0,T]$ the symbol x_t denotes the function on $[-h,0]$ defined by:

$$x_{tt}(s)=(t+s);s\in[-h,0],h>0$$

Consider the system defined by the integro-differential equation:

$$x(t)=A(t)x(t)+B(t)x(t-1)+\int_{-1}^{0}K(t,s)x(t+s)ds---------(2.1)$$ and its perturbation

$$x(t)=A(t)x(t)+B(t)x(t-1)+\int K(t,\ s)x(t+s)ds+f(t+x_t)--(2.2)$$

The vector function *X* has its values in *E*. The matrix function *A, B* have appropriate dimensions and are assumed to be continuous on $J,J\in E$. The matrix function *K* is also an n x n matrix that is continuous on

$$JX[-1,0],let\ h>0$$

Let ϕ be an n-vector function which is continuous on $[-1,\ 0]$. Then there exists a unique solution of system *(2.1)* on *J* satisfying $X(t)=\phi\ \ for\ t\in[-1m0]$.

This solution is given as

$$x(t)=X(t,0)\ \ \phi(0)+\int_{-1}^{0}\int_{0}^{w+1}X(t,s)\ K(s,w-s)dsdw$$

The solution of *(2.2)* is given as;

$$x(t)=X(t,0)\phi(0)+\int_{-1}^{0}\int_{0}^{w+1}X(t,s)K(s,w-s)dsdw+\int_{0}^{1}X(t,s)\ f(t,x_s)ds$$

where $X(t,s)$ is an *n x n* matrix function satisfying

$$\frac{dx(t,s)}{dt}=A(t)X(t,s)+B(t)X(t-1,s)+\int_{-1}^{0}K(t,w)X(t+w,s)dw$$

For $0\le s\le t\le t_1$ such that $X(t,t)=1,$ the identity matrix and $X(t,s)=0\ for t<s$.

2.1 Preliminary Results of Periodic Systems

Definitions 2.1 - Periodic system

Consider the general non-linear ordinary differential equation given by:

$x(t) = F(t, x(t)) ------------------------ (2.3)$

With $F : E \, X \, E^n \to E^n$ continuous

If for all $(t,x) \in EXE^n$ *and* $T > 0$ the relation

$f(t,x) = f(t+t,x) --------------------- (2.4)$

holds, the system (2.3) is said to be T-periodic. In general, such a system does not admit a periodic solution unless some conditions are imposed. It is therefore natural and useful for application to see that such a system admits solutions of the form

$x(t+T) = x(t) \;\; for\, all\, t \in E; T > 0$

Definition 2.2 – Periodic Solution

The solution X of (2.3) defined on E such that

$x(t+T) = x(t) ------------------------ (2.5)$

For all *t* in *E* is called *T*-periodic or harmonic solution. Another natural solution is the existence for some integer $k > 1$ of solution *X* defined on *E* and such that

$x(t+k) = x(t) \; for\, all\, t \in E$

This solution *X* defined on *E* such that

$x(t+kT) = x(t) ---------------------- (2.6)$

For some integer $k > 1$ is called sub-harmonic solutions of order *k* of the *T*-periodic solution.

Definition 2.3

$x = A(t)x -------------------------- (2.7)$

Where $A(t)$ is a continuous $n \times n$ matrix function of *t* such that

$A(t+w) = A(t) ---------------------- (2.8)$

For some constant $w > 0;$ then (2.8) is called a periodic system and *w* is called the periodic of $A(t)$.

The following theorem is a fundamental result for system (2.7) and it conveys that; the fundamental matrix solution of a periodic homogenous system is also periodic and has an exponential characterisation.

Theorem 2.1

Let $\phi = \phi(t)$ be a fundamental matrix of (2.7) then $\psi = \psi(t)$ defined by

$\varphi(t) = \varphi(t+w)(-\alpha < t < \alpha)$ is also a fundamental matrix for (2.7). Corresponding is every such fundamental matrix, there exist a periodic non-singular $n \times n$ matrix equal to $\rho(t)$ with period *w*, and a constant matrix *R* such that

$$\phi(t) = \rho(t)e^{n} ------------------------(2.9)$$

Proof

Let $\phi = \phi(t)$ be fundamental matrix of (2.7) and then

$$\phi(t+w) = A(t+w)\phi(t+w)$$

$$\Rightarrow \phi(t+w) = A(t)\phi(t+w) \text{ - by periodicity of } A(t)$$

$$\psi(t) = A(t)\psi(t)$$

Thus $\psi(t) = \phi(t+w)$ is a solution matrix of (2.7). It is a fundamental matrix of (2.6) since the determinant of ψ at t_0 or

$$\det \psi(t)\big|_{t=0} = \det \phi(t+w)\big|_{t=0} = \det \phi(t) \neq 0$$

clearly, there exists a non-singular constant $n \times n$ matrix C such that

$$\phi(t+w) = \phi(t)C ----------------------(2.10)$$

Furthermore, there exists a constant matrix R such that

$$C = e^{wR} ---------------------------(2.11)$$

(2.11) has been proved already by Coddington and Levison. Combining (2.10) and (2.11), we have that

$$\phi(t+w) = \phi(t)e^{wR} --------------------(2.12)$$

Now let $\rho = \rho(t)$ be defined by

$$\rho(t) = \phi(t)e^{tR} ----------------------(2.13)$$

Then by (2.11)

$$\rho(t+w) = \phi(t+w)e^{-(t+w)r}$$

$= \phi(t)e^{-IR}\rho(t)$

Therefore $\rho(t)$ is periodic. Since $\phi(t)$ and e^{tR} are non-singular, so also is $\rho(t)$

Theorem 2.2 – Consider $x(t) = A(t)x + B(t)\,x(t-h)$

The proof of this is straight forward from the result of Theorem 2.1.

Proposition 2.1

If $t_0 \in E$ and *X* is a solution of (2.3), defined by $[t_0, t_0 + T]$ such that

$$x(t_0 + T) = x(t_0) -------------------(2.14)$$

Then the function *x* can be extended to *E* by periodicity and the corresponding extension will be a *T*-periodic solution.

Proof

If $t \in E; n \in N$ such that

$$t \in [t_0 + (n-1)T_0 + nT]$$

Hence $t-(n-1)T \in [t_0, t_0 + T]$

Define the extension of $x(t)$ by

$$\bar{x}(t) = x(t-(n-1)t) -------------------(2.15)$$

with $t-(n-1)t \in [t_0, t_0 + T]$

Therefore *x(t)* is an extension of *X* to *E* by periodicity. Also *x(t)* coincides with $\bar{x}(t)$ on $[t_0, t_0 + T]$. Next, we show that $x(t)$ satisfies (2.3) on *E*.

$$\dot{x}(t) = \frac{d}{dt}x\big(t-(n-1)T\big) = f\big(t_0 - (n-1)T, x(t-(n-1)T\big)$$

$= f\big(t-(n-1)T, x(t)\big)$ by (2.15)

$= f\big(t, x(t)\,\big)$ by (2.3) for all $t \in E$

Thus $X(t)$ satisfies (2.2) on *E*. In particular, if we can find a solution of (2.2) such that

$$x(t_0) = (t_0 + T);\ t_0 = c,$$

$$\Rightarrow x(0) = x(T) ---------------------- (2.16)$$

Then the extension by periodicity of *k* solution furnishes us with a *T*-periodic solution of (2.7)

2.2 Conditions for the existence and uniqueness of periodic solution for delay equation.

Hale ( ) has provided continuity conditions on *f* for the existence and uniqueness of the system.

$\dot{x} = f(t_1 x_t)$. Ukwu ( ) has established that the non criticality of the homogenous part of a system guarantees the existence of a periodic solution for the system. Burton, T. A. ( ), Cheloan D. N. ( ) and ( ) have also contributed on the subject.

3.0 Variation of Constants Formula

Consider the system given by

$$x(t) = Ax(t) + x(t-1) + \int_{-1}^{0} k(t,s)\, x(t+x)ds --------(3.1)$$

$$x(t) = \phi(0) = x(t)\ for\, t \in [-1, 0]$$

and its perturbation

$$x(t) = Ax(t) + Bx(t-1) + \int_{-1}^{0} K(t,s)\, x(t+s)ds + f(t,x) --- (3.2)$$

$$x(0) = \phi(0) = x(t)\ for\, t \in [-1, 0]$$

Where A and B are continuous $n \times n$ matrix functions of *t*.

$f:[0,T]\, XE^n \to E^n$ is also continuous

$$x(t) = A(t)x(t) + Bx(t-1) ---------------- (3.3)$$

The solution of system (3.3) as reported in Ukwu ( ) is given by

$$x(t) = X(t,s)\phi ---------------------- (3.4)$$

Where *x(t,s)* is the fundamental matrix of (3.3).

Using this result, we obtain the variation of constant formula for system (3.1)

$$x(t) = X(t,0)\psi(0) + \int_{-1}^{0} X(t,s)\left[\int_{-1}^{0} K(t,s)\, x(t+s)ds\right]dt$$

We also obtain the variation of constant formula or system (3.2) given by:

$$x(t) = X(t,0)\phi(0) + \int_{-1}^{0} X(t,s)\left[K(t,s)\,x(t,s)\,x(t+s)ds\right]dt + \int_{-1}^{0} X(t,s)\ f(t,x)\,ds ----(3.5)$$

Suppose $t = T$, equation (3.5) becomes

$$x(T) = X(t,0)\left\{\phi(0) + 0\int_{-1}^{0} X(0,w)\left[\int_{-1}^{0} K(t,s)\,x(t+s)ds\right]\right\}dw\ +$$

$$\int_{-1}^{0} X(0,s)\ f(s,x_s)ds ----------------(3.6)$$

Here, we assume that system (3.3) is non-critical and that, is to say that

$$\phi(0) = x(T) = C ----------------------------------(3.7)$$

Where C is a constant, as disclosed earlier by Ukwu (   ). Using condition (3.7) in (3.4), we have

$$x(T) = X(t,0)\phi(0) \Rightarrow C = X(t,0)C ------------------------(3.8)$$

At $t = T$ and using (3.8) in (3.6), we have

$$X(T) = X(T,0)\left[C + \int_{-1}^{0} X(0,w)\int_{-1}^{0} k(w,s)\ \phi(s)ds\right]dw + \int_{-1}^{0} X(0,s)\ f(s,x_s)ds ----(3.9)$$

$$\Rightarrow C = X(T,0)C + X(T,0)\left[\int_{-1}^{0} X(0,w)\int_{-1}^{0} K(w,s)\phi(s)ds\right] + \int_{-1}^{0} X(0,s)\ f(s,x_s)ds$$

Where $x(t+s) = \phi(s)$. From equation (3.9), we have

$$C\left[I - X(T,0)\right] = \int_{-1}^{0} K(t,w)\left[\int_{-1}^{0} K(w,s)\phi(s)ds\right]dw + \int_{-1}^{0} X(T,s)\ f(s,x_s)ds ------(3.10)$$

Evidently; $[I - X(T,0)]$ is non-singular and hence invertible so C can be written as

$$C = \left[I - X(0,T)^{-1}\right]\left[t_0 X(T,0)\left\{\int_{-1}^{0} K(w,s)\phi(s)\,ds\right\}dw + \int_{-1}^{0} X(T,s)\ f(s,x_s)ds\right] --(3.11)$$

C is said to be well-defined since $[I - X(t,0)]$ is non-singular.

Following the non-singularity of the fundamental matrix and X, and the identity matrix I, then substituting C into (3.5) with $C = \phi(0),$ we have;

$$x(t) = X(t,0)\left\{[t - X(T,0)]^{-1}\int_0^T X(T,s)\left[\int_{-1}^0 K(T,w)\phi(s)ds\right]dw + \int_{-1}^0 X(T,s)\, f(s,x_s)ds\right\}$$
$$+\int_{-1}^0 X(t,s)\int_{-1}^0 K(t,w)\phi(s)ds)ds + \int_{-1}^0 X(t,s)\, f(s,x_s)ds ------------- (3.12)$$

We now expand equation (3.12) to obtain the following expression

$$x(t) = X(t,0)\left\{[I - X(T,0)]^{-1}\int_{-1}^0 X(T,s)\left[\int_{-1}^0 X(t,w)\phi(s)\,ds\right]dw\right.$$
$$\left.+\int_{-1}^0 X(T,S)\, f(s,x_s)ds\right\} + x(t,0)[I - X(T,0)]^{-1}$$
$$\int_{-1}^0 X(T,s)\left[\int_{-1}^0 K(T,w)\phi(s)ds\right]dw + \int_{-1}^0 X(T,s) f(s,x_s)ds$$
$$+\int_{-1}^0 X(t,s)\left[\int_{-1}^0 K(s,w)\phi(s)\right]dw$$
$$+\int_{-1}^0 X(t,s)\, f(s,x_s)ds = \int_{-1}^0 G(t,s)\, f(s,x,)ds ------------(3.13)$$

4.0 The Green Matrix

To obtain Green Matrix, we further expand (3.13) into:

$$G = \begin{cases} X(t,0)[I - X(T)^t](s)\left[\int_{-1}^0 k(T,w)\phi(s)ds\right]dw \\ + X(T,s)\, f(x,x_s)ds + X(t,s)\left[\int_{-1}^0 k(s,w)\phi(s)ds\right]dw \\ + X(t,s)\, f(s,x_s)ds; 0 \le s \le t < T \\ X(t,0)[I - X(T,0)]^{-1} X(t,s)\left[\int_{-1}^0 k(T,w)\phi(s)ds\right]dw \\ + X(T,s)\, f(s,x_s)ds; 0 < t < s < T \end{cases}$$

Evidently, we have the following properties of the Green Matrix

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(i) $G(t,s)$ is differentiable almost everywhere for $t \in [0,T]-\{s\}$ and

$$\frac{dG}{dt} = AG(t) + BG(t-1)$$

(ii) $G(t,s)$ is continuous except at $t = sG(s+0,s) - G(s-0,s) = 1$

Hence, *G* is bounded and almost continuous everywhere.

(iii) $G(0,s) = G(t,s)$

Banach fixed point theorem

Let $T.Cx \to Cy$ be a mapping from a bounded set to another bounded set if *T* is a contraction mapping. It has a fixed point in *Cr* that is for some $x \in Cr; and\, Ts = x$.

Main results

Existence and uniqueness for the perturbation of integro-differential equation with a non-critical part

Existence and uniqueness theorem

In this section, we give sufficient conditions on *f* to ensure the existence of periodic solution for the system (3.1).

Theorem 3.1

Consider the integro-differential system

$$x(t) = Ax(t) + (t-1) + \int_{-0}^{0} k(t,s)ds + f(t,s);$$

$$x(t) = \phi(0) \; for\, t \in [-1,0]$$

with the linear homogenous differential part (3.3) being non-critical, that is the solution is such that for our system,

$$X(0) = X(T)$$

(i) *f* is Lipschitzian with respect to *X* for all $t \in [0,T]$. That is to say that there exist

$\lambda > 0, R > 0$ such that

$$|f(t,x) - f(t,y)| \le \lambda |x_t - y_t|; \le \lambda |x_t - y_t|$$

$$|x_t| \le r; |y_t| \le R$$

Also, there exist $\gamma < 1 \; and \; \delta < (1-\gamma)R$

Where $\gamma = \lambda Sup\int_0^T |G(t,s)ds;|\gamma < \infty$ and

$$d = \lambda Sup\int_0^t |G(t,s)\|f(s,0)|ds$$

for $t \in [0,T]$

Then the boundary value problem has a unique *T*-periodic solution.

Proof:

Let C be the Banach space of all continuous functions

$:[-h,T] \to E^n$ such that $x(0) = x(T)$

Let C_R be a compact ball in C of radius R, for sufficiently large R, C_R can be chosen non-empty, we now define an operator.

$L = C_R \to C_R$ given as:

$$L[x(t)] = \int_0^T G(t,s)\, f(s,x_s)ds$$

And then pure that L is a contraction, we first prove that L is well defined, that is for $X \in C_R$

$L(x) \in C_R; Let\; X \in C_R,$

$$|L(x(t))\|\int_0^T G(t,s)\, f(s,x_s)ds|$$

$$= \left|\int_0^T G(t,s)[f(s,x_s)\, f(s,0) + f(s,x_s)]ds\right|$$

$$\leq \int_0^T |G(t,S)\|f(s,x_s) - f(s,0)| + \int_0^T |G(t,s)\|f(s,xs)|ds$$

$$\leq \lambda Sup\int_0^T |G(t,s)|ds\|x\|_c + \delta$$

$t \in [0,T]$

$\gamma \|x\|_c + \delta \leq R$

Clearly, $\|x\| < R$ shows that

$L(x)(t) \in C_R$ hence *L* is well-defined.

Next, we show that *L* is a contradiction on C_R.

$$|L(x)(t) - L(y)(t)| < \int_0^T |G(t,s)\| f(s,x_s) - f(s,y_s)| ds$$

$$\leq \int_0^T |G(t,s)\| x_s - y_s| ds$$

Take the supreme of both sides over $[0,T]$, you get

$$L[x_t] - L[y_t] \leq \lambda Sup \int_0^T |G(t,s) ds\| x_s - y_s|$$

$$< Y \| X_s - Y_s \| \; for \; X_s Y_s \in C_R$$

Since $Y < 1$, *L* is shown to be a contraction. And by Banach's fixed point theorem or the contraction mapping principle,

$L(x)(t) = x(t)$ - a fixed point which is the required solution for the boundary value problem. This completes the existence theorem.

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